7.2 Finding Complex Solutions of Polynomial Equations Date: Key

Investigating the Number of Complex Zeros of a Polynomial Function

Consider the polynomial $p(x) = x^3 + 8x^2 + 21x + 18 = (x+2)(x+3)^2$

What are the factors of *p(x)*? What is different about the factors?

$(x+2)$ and $(x+3)$; $(x+3)$ occurs twice

Multiplicity:	number of times a factor occurs

In this exploration, you will use algebraic methods to investigate the relationship between the degree of a polynomial function and number of zeros that it has.

For each of the functions below: find all zeros and include any multiplicities greater than 1.

A) $p(x) = x^3 + 7x^2$

$p(x) = x^2(x+7)$

$\boxed{x=0 \text{ mult}=2}$ $\boxed{x=-7}$

B) $p(x) = x^3 - 64. = (x)^3 - (4)^3$

$p(x) = (x-4)(x^2+4x+16)$

$\boxed{x=4}$ one real

$x = \dfrac{-4 \pm \sqrt{16-4(1)(16)}}{2(1)}$

$x = \dfrac{-4 \pm \sqrt{-48}}{2} = \dfrac{-4}{2} \pm \dfrac{4i\sqrt{3}}{2}$

$\boxed{x = -2 \pm 2i\sqrt{3}}$

2 non-real

C) $p(x) = x^4 + 3x^3 - 4x^2 - 12x$

$p(x) = x(x^3+3x^2-4x-12)$

$p(x) = x[x^2(x+3) - 4(x+3)]$

$p(x) = x(x+3)(x^2-4)$

$p(x) = x(x+3)(x+2)(x-2)$

$\boxed{x = 0, -3, -2, 2}$

D) $p(x) = x^4 - 16$

$= (x^2)^2 - (4)^2$

$= (x^2+4)(x^2-4)$

$= (x^2+4)(x+2)(x-2)$

$\boxed{x = -2, 2}$

Two real

$x^2+4=0$

$x^2=-4$

$\boxed{x = \pm 2i}$

Two non-real

E) $p(x) = x^4 + 5x^3 + 6x^2 - 4x - 8$ use RZT

1	1	5	6	-4	-8
	↓	1	6	12	8
	1	6	12	8	0

$p(x) = (x-1)(x^3+6x^2+12x+8)$

-2	1	6	12	8
	↓	-2	-8	-8
	1	4	4	0

$p(x) = (x-1)(x+2)(x^2+4x+4)$

$p(x) = (x-1)(x+2)(x+2)(x+2)$

$p(x) = (x-1)(x+2)^3$

$x=1$
$x=-2$ (mult. 3)

Complete the table to summarize your results from A – E.

Polynomial Function in Standard Form	Polynomial Function in Factored Form	Real Zeros and their Multiplicities	Non-real Zeros and their Multiplicities
$p(x) = x^3 + 7x^2$	$p(x) = x^2(x+7)$	$x = 0$ (mult 2) $x = -7$	none
$p(x) = x^3 - 64$	$p(x) = (x-4)(x^2+4x+16)$	$x = 4$	$-2 - 2i\sqrt{3}$ $-2 + 2i\sqrt{3}$
$p(x) = x^4 + 3x^3 - 4x^2 - 12x$	$p(x) = x(x+3)(x+2)(x-2)$	$x = 0, -3, -2, 2$	none
$p(x) = x^4 - 16$	$p(x) = (x-2)(x+2)(x^2+4)$	$x = 2, -2$	$-2i$, $2i$
$p(x) = x^4 + 5x^3 + 6x^2 - 4x - 8$	$p(x) = (x-1)(x+2)^3$	$x = 1$ $x = -2$ (mult 3)	none

Reflect.

1. For each function, count the **number of unique zeros**, both real and non-real. How does the number of unique zeros compare with the degree of the polynomial?

The number of unique zeros is less than or equal to the degree

2. Count the **total number of zeros** for each function. How does the total number of zeros compare with the degree?

The total number of zeros is the same as the degree

3. Describe the apparent relationship between the degree of a polynomial function and the number of zeros it has.

The number of zeros a polynomial function has is equal to its degree

Applying the Fundamental Theorem of Algebra to Solving Polynomial Equations

Learning Target C: *I can use the Fundamental Theorem of Algebra to find all roots of a polynomial equation.*

The Fundamental Theorem of Algebra
Every polynomial function of degree $n \geq 1$ has at least one zero, where a zero may be a complex number. **Corollary:** Every polynomial function of degree $n \geq 1$ has exactly n zeros, including multiplicities.

Because the zeros of a polynomial function p(x) give the roots of the equation $p(x) = 0$, the theorem and its corollary also extend to finding all roots of a polynomial equation.

Solve the polynomial equation by finding all roots.

A) $2x^3 - 12x^2 - 34x + 204 = 0$

$x^3 - 6x^2 - 17x + 102 = 0$

$(x^3 - 6x^2) + (-17x + 102) = 0$

$x^2(x-6) + -17(x-6) = 0$

$(x^2-17)(x-6) = 0$

$\boxed{x = \sqrt{17}, -\sqrt{17}, 6}$

B) $x^4 - 6x^2 - 27 = 0$

$(x^2)^2 - 6(x^2) - 27 = 0$

$(x^2 - 9)(x^2 + 3) = 0$

$(x+3)(x-3)(x^2+3) = 0$

$x^2 = -3$

$x = \pm i\sqrt{3}$

$\boxed{x = -3, 3, i\sqrt{3}, -i\sqrt{3}}$

C) $8x^3 - 27 = 0$

$(2x)^3 - (3)^3 = 0$

$(2x-3)(4x^2+6x+9) = 0$

$\boxed{x = \frac{3}{2}}$

$x = \frac{-6 \pm \sqrt{36 - 4(4)(9)}}{2(4)}$

$x = \frac{-6 \pm \sqrt{-108}}{8} = \frac{-6}{8} \pm \frac{6i\sqrt{3}}{8}$

$-36 \cdot 3$

$\boxed{x = -\frac{3}{4} \pm \frac{3i\sqrt{3}}{4}}$

D) $p(x) = x^4 - 13x^3 + 55x^2 - 91x$

± 1
± 7
± 13
± 91

$0 = x(x^3 - 13x^2 + 55x - 91)$

$$\begin{array}{r|rrrr} 7 & 1 & -13 & 55 & -91 \\ & \downarrow & 7 & -42 & 91 \\ \hline & 1 & -6 & 13 & \boxed{0} \end{array}$$

$0 = x(x-7)(x^2 - 6x + 13)$

$\boxed{\begin{array}{l} x = 0 \\ x = 7 \end{array}}$

$x = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2}$

$x = \frac{6 \pm \sqrt{-16}}{2} = \frac{6}{2} \pm \frac{4i}{2}$ 3

$\boxed{x = 3 \pm 2i}$

Writing a Polynomial Function from Its Zeros

Learning Target D: *I can write a polynomial function given its zeros.*

Sometimes we have roots of quadratic or polynomial equation that are irrational or complex.

For example, we could have the pair of ***complex roots***:

2 + 14i and 2 – 14i, which we can write as $2 \pm 14i$

...or a pair of ***irrational roots***:

$1+\sqrt{7}$ and $1-\sqrt{7}$, which we can write as $1 \pm \sqrt{7}$

★ These pairs are called complex conjugates and irrational conjugates.

Irrational Root Theorem

If a polynomial $p(x)$ has rational coefficients and $a + b\sqrt{c}$ is a root of the equation $p(x) = 0$, where a and b are rational and $\sqrt{c}$ is irrational, then $a - b\sqrt{c}$ is also a root of $p(x) = 0$.

Complex Conjugate Root Theorem

If $a + bi$ is an imaginary root of a polynomial equation with real-number coefficients, then $a - bi$ is also a root.

Write the polynomial function in standard form with the least degree and a leading coefficient of 1 that has the given zeros.

A) 5 and $3 + 2\sqrt{7}$

$p(x) = (x + (3+2\sqrt{7}))(x-(3+2\sqrt{7}))(x-5)$

work on next page

$\boxed{p(x) = x^3 - 11x^2 + 11x + 95}$

B) 2, 3, and 1 – i

$p(x) = (x-2)(x-3)(x-(1-i))(x-(1+i))$

$\boxed{p(x) = x^4 - 7x^3 + 18x^2 - 22x + 12}$

A 5 and $3 + 2\sqrt{7}$

Because irrational zeros come in conjugate pairs, $3 - 2\sqrt{7}$ must also be a zero of the function. Use the 3 zeros to write a function in factored form, then multiply to write it in standard form.

$$p(x) = \left[x - \left(3 + 2\sqrt{7}\right)\right]\left[x - \left(3 - 2\sqrt{7}\right)\right](x - 5)$$

Multiply the first two factors using FOIL. $= \left[x^2 - \left(3 - 2\sqrt{7}\right)x - \left(3 + 2\sqrt{7}\right)x + \left(3 + 2\sqrt{7}\right)\left(3 - 2\sqrt{7}\right)\right](x - 5)$

Multipy the conjugates. $= \left[x^2 - \left(3 - 2\sqrt{7}\right)x - \left(3 + 2\sqrt{7}\right)x + (9 - 4 \cdot 7)\right](x - 5)$

Combine like terms. $= \left[x^2 + \left(-3 + 2\sqrt{7} - 3 - 2\sqrt{7}\right)x + (-19)\right](x - 5)$

Simplify. $= \left[x^2 - 6x - 19\right](x - 5)$

Distributive property $= x(x^2 - 6x - 19) - 5(x^2 - 6x - 19)$

Multiply. $= x^3 - 6x^2 - 19x - 5x^2 + 30x + 95$

Combine like terms. $= x^3 - 11x^2 + 11x + 95$

The polynomial function is $p(x) = x^3 - 11x^2 + 11x + 95$.

2, 3 and $1 - i$

Because complex zeros come in conjugate pairs, $\underline{1 + i}$ must also be a zero of the function.

Use the 4 zeros to write a function in factored form, then multiply to write it in standard form.

$$p(x) = \left[x - (1 + i)\right]\left[x - \left(\boxed{1 - i}\right)\right](x - 2)(x - 3)$$

Multiply the first two factors using FOIL. $= \left[x^2 - (1 - i)x - \left(\boxed{1 + i}\right)x + (1 + i)(1 - i)\right](x - 2)(x - 3)$

Multipy the conjugates. $= \left[x^2 - (1 - i)x - (1 + i)x + \left(1 - \left(\boxed{-1}\right)\right)\right](x - 2)(x - 3)$

Combine like terms. $= \left[x^2 + (-1 + i - 1 - i)x + 2\right](x - 2)(x - 3)$

Simplify. $= \left(\boxed{x^2 - 2x + 2}\right)(x - 2)(x - 3)$

Multipy the binomials. $= (x^2 - 2x + 2)\boxed{(x^2 - 5x + 6)}$

Distributive property $= x^2(x^2 - 5x + 6)\boxed{-2x}(x^2 - 5x + 6) + 2(x^2 - 5x + 6)$

Multipy. $= (x^4 - 5x^3 + 6x^2) + (-2x^3 + 10x^2 - 12x) + (2x^2 - 10x + 12)$

Combine like terms. $= \boxed{x^4 - 7x^3 + 18x^2 - 22x + 12}$

The polynomial function is $p(x) = \underline{x^4 - 7x^3 + 18x^2 - 22x + 12}$.