11.2 Vectors in Space

Objectives

- Find the component forms of the unit vectors in the same direction of, the magnitudes of, the dot products of, and the angles between vectors in space.
- Determine whether vectors in space are parallel.

Vectors in Space

If **v** is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, as shown in Figure 11.9, then the **component form** of **v** is produced by subtracting the coordinates of the initial point from the corresponding coordinates of the terminal point

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

$$= \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle.$$

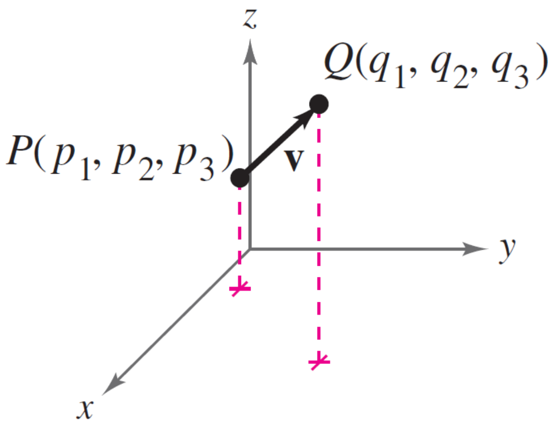


Figure 11.9

Example 1

Find the component vector of $\boldsymbol{v}$.

Initial Point $(1, -2, 6)$ Terminal Point $(3, 2, -4)$

$$v = \langle 3 - 1, 2 - -2, -4 - 6 \rangle$$

$$v = \langle 2, 4, -10 \rangle$$

Vectors in Space

Vectors in Space

1. Two vectors are **equal** if and only if their corresponding components are equal.

2. The **magnitude** (or **length**) of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$.

3. A **unit vector u** in the direction of $\mathbf{v}$ is $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}, \quad \mathbf{v} \neq \mathbf{0}$.

4. The **sum** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle.$$ Vector addition

5. The **scalar multiple** of the real number c and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is

$$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle.$$ Scalar multiplication

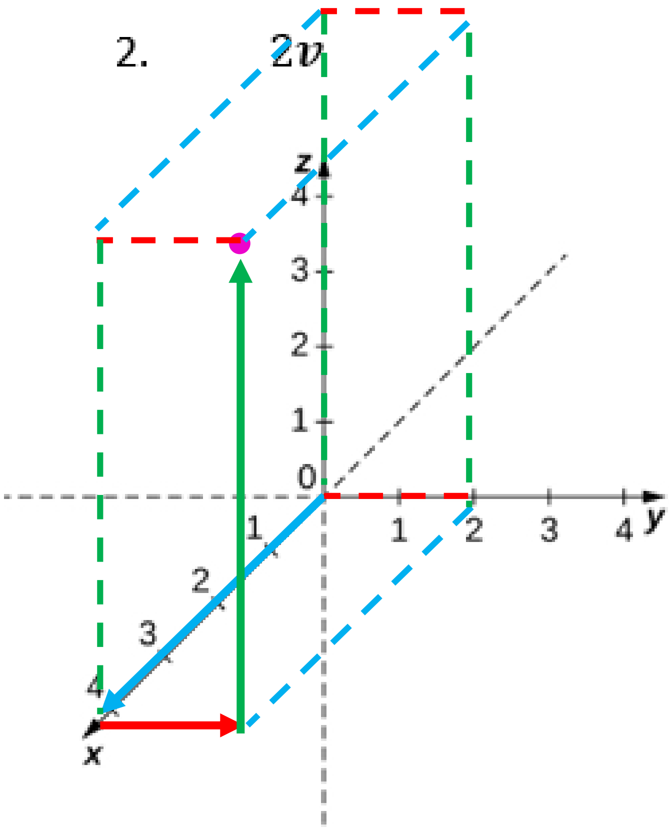
6. The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$ Dot product

Examples 2 and 3

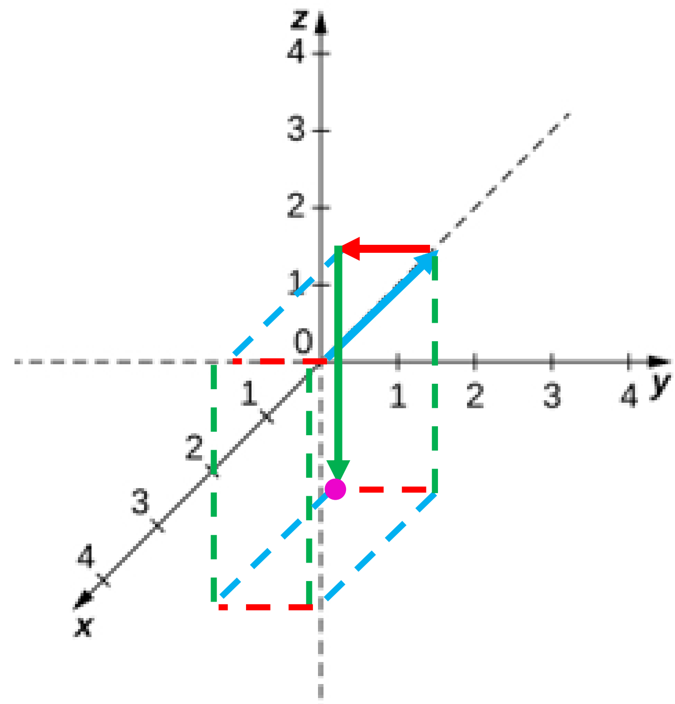
Sketch the scalar multiple of $v = \langle 2, 1, 3 \rangle$

2. $2v$



$2v = \langle 4, 2, 6 \rangle$

3. $-v$



$-v = \langle -2, -1, -3 \rangle$

Example 4

Find the vector $\boldsymbol{z}$ if $\boldsymbol{a} = \langle -2,4,1\rangle$, $\boldsymbol{s} = \langle 3,3,0\rangle$, and $\boldsymbol{u} = \langle 3,3,0\rangle$.

a. $\boldsymbol{z} = \boldsymbol{a} + \boldsymbol{s} + \boldsymbol{u}$

$$z = \langle -2 + 3 + 3, 4 + 3 + 3, 1 + 0 + 0\rangle$$

$$z = \langle 4, 10, 1\rangle$$

b. $\boldsymbol{z} = 2\boldsymbol{a} - \boldsymbol{u}$

$$z = \langle -4 - 3, 8 - 3, 2 - 0\rangle$$

$$z = \langle -7, 5, 2\rangle$$

c. $\boldsymbol{z} = \boldsymbol{a} + \boldsymbol{3s} - \frac{1}{3}\boldsymbol{u}$

$$z = \langle -2 + 9 - 1, 4 + 9 - 1, 1 + 0 - 0\rangle$$

$$z = \langle 6, 12, 1\rangle$$

Example 5

Find the magnitude of $\boldsymbol{v}$.

a. $\boldsymbol{v} = \langle -2, 0, -5 \rangle$

$$\|v\| = \sqrt{(-2)^2 + (0)^2 + (-5)^2}$$

$$\|v\| = \sqrt{4 + 0 + 25}$$

$$\|v\| = \sqrt{29}$$

b. $\boldsymbol{v} = -i - 4j + 3k$

$$\|v\| = \sqrt{(-1)^2 + (-4)^2 + (3)^2}$$

$$\|v\| = \sqrt{1 + 16 + 9}$$

$$\|v\| = \sqrt{26}$$

Example 6

Find the unit vector of $\boldsymbol{v}$.

a. $\boldsymbol{v} = \langle -2,1,-5\rangle$

$$\frac{1}{\sqrt{(-2)^2+(1)^2+(-5)^2}}\langle -2,1,-5\rangle$$

$$\frac{1}{\sqrt{4+1+25}}\langle -2,1,-5\rangle$$

$$\frac{1}{\sqrt{30}}\langle -2,1,-5\rangle$$

OR

$$\langle \frac{-2}{\sqrt{30}},\frac{1}{\sqrt{30}},\frac{-5}{\sqrt{30}}\rangle$$

b. $\boldsymbol{v} = -i - 4j + 3k$

$$\frac{1}{\sqrt{26}}\langle -1,-4,3\rangle$$

OR

$$\langle \frac{-1}{\sqrt{26}},\frac{-4}{\sqrt{26}},\frac{3}{\sqrt{26}}\rangle$$

Example 7

Find the dot product of $\boldsymbol{a} = \langle -2,4,1 \rangle$ and $\boldsymbol{s} = \langle 3,3,0 \rangle$.

$$\langle -2, 4, 1 \rangle \cdot \langle 3, 3, 0 \rangle = -2(3) + 4(3) + 1(0)$$

$$= -6 + 12 + 0$$

$$= 6$$

Note that the dot product of two vectors is a real number, not a vector.

Parallel, Orthogonal or Neither

To determine if vectors are parallel compare the slopes of each component, if the slopes are all the same, then the vectors are parallel.

To determine if vectors are orthogonal, find the dot product of the two vectors. If the dot product is zero then the vectors are orthogonal.

If both tests above prove to be wrong then the vectors are neither.

Example 8

Determine if the vectors are orthogonal, parallel, or neither.

a. $\boldsymbol{a} = \langle -1, 4, 3 \rangle$ and $\boldsymbol{s} = \langle 2, -8, -6 \rangle$

$$\frac{s}{a} = \left\langle \frac{2}{-1}, \frac{-8}{4}, \frac{-6}{3} \right\rangle \qquad \frac{s}{a} = \langle -2, -2, -2 \rangle$$

The slopes of all three components are the same so the vectors are parallel

b. $\boldsymbol{a} = 2i - 3j + k$ and $\boldsymbol{s} = -i - j - k$

$$\frac{s}{a} = \left\langle \frac{-1}{2}, \frac{-1}{-3}, \frac{-1}{1} \right\rangle$$

The slopes of all three components are NOT the same so now find the dot product.

$$2(-1) + (-3)(-1) + (1)(-1)$$

$$-2 + 3 - 1$$

$$0$$

The vectors are orthogonal

Example 9

Are the points $(-2,7,4), (-4,8,1), \text{and } (0,6,7)$ Collinear?

$\langle -4 - -2, 8 - 7, 1 - 4 \rangle$

$\langle 0 - -2, 6 - 7, 7 - 4 \rangle$

$\langle -2, 1, -3 \rangle$

$\langle 2, -1, 3 \rangle$

Slopes= $\left\langle \frac{2}{-2}, \frac{-1}{1}, \frac{3}{-3} \right\rangle$

Slopes= $\langle -1, -1, -1 \rangle$

The slopes of all three components are the same so the vectors are parallel. Therefore all three points are collinear.

Vectors in Space

Angle Between Two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$.

If the dot product of two nonzero vectors is zero, then the angle between the vectors is 90° (recall that cos 90° = 0). Such vectors are called **orthogonal.**

For instance, the standard unit vectors **i**, **j**, and **k** are orthogonal to each other.

Example 10

Find the angle between the two vectors,

$\boldsymbol{a} = \langle -1,3,0 \rangle$ and $\boldsymbol{s} = \langle 1,2,-1 \rangle$

$$\theta = cos^{-1}\left(\frac{-1(1) + (3)(2) + (0)(-1)}{\left(\sqrt{(-1)^2 + (3)^2 + (0)^2}\right)\left(\sqrt{(1)^2 + (2)^2 + (-1)^2}\right)}\right)$$

$$\theta = cos^{-1}\left(\frac{-1 + 6 + 0}{\left(\sqrt{1 + 9 + 0}\right)\left(\sqrt{1 + 4 + 1}\right)}\right)$$

$$\theta = cos^{-1}\left(\frac{5}{\left(\sqrt{10}\right)\left(\sqrt{6}\right)}\right)$$

$$\theta = 49.8°$$