11.3

The Cross Product of Two Vectors

Objectives

- Find cross products of vectors in space.

- Use geometric properties of cross products of vectors in space.

The Cross Product

Many applications in physics, engineering, and geometry involve finding a vector in space that is orthogonal to two given vectors.

In this section, you will study a product that will yield such a vector. It is called the **cross product,** and it is conveniently defined and calculated using the standard unit vector form.

Definition of Cross Product of Two Vectors in Space

Let

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

and

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

be vectors in space. The **cross product** of **u** and **v** is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

The Cross Product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Put **u** in Row 2.

Put **v** in Row 3.

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{k}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Example 1a

If $\boldsymbol{a} = 2i + 3j - k$, $\boldsymbol{s} = -i + 4j + 2k$, and $\boldsymbol{u} = 2i + j$, find:

a. $\boldsymbol{a} \times \boldsymbol{s}$

$$a \times s = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ -1 & 4 & 2 \end{vmatrix}$$

$$a \times s = i\begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} - j\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + k\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix}$$

$$a \times s = (6 - -4)i - (4 - 1)j + (8 - -3)k$$

$$a \times s = 10i - 3j + 11k$$

Example 1b

If $\boldsymbol{a} = 2i + 3j - k$, $\boldsymbol{s} = -i + 4j + 2k$, and $\boldsymbol{u} = 2i + j$, find:

b. $\boldsymbol{s} \times \boldsymbol{u}$

$$s \times u = \begin{vmatrix} i & j & k \\ -1 & 4 & 2 \\ 2 & 1 & 0 \end{vmatrix}$$

$$s \times u = i \begin{vmatrix} 4 & 2 \\ 1 & 0 \end{vmatrix} - j \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} + k \begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix}$$

$$s \times u = (0 - 2)i - (0 - 4)j + (-1 - 8)k$$

$$s \times u = -2i + 4j - 9k$$

Example 1c

If $\boldsymbol{a} = 2i + 3j - k$, $\boldsymbol{s} = -i + 4j + 2k$, and $\boldsymbol{u} = 2i + j$, find:

c. $\boldsymbol{s} \times \boldsymbol{a}$

$$s \times a = \begin{vmatrix} i & j & k \\ -1 & 4 & 2 \\ 2 & 3 & -1 \end{vmatrix}$$

$$s \times a = i\begin{vmatrix} 4 & 2 \\ 3 & -1 \end{vmatrix} - j\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} + k\begin{vmatrix} -1 & 4 \\ 2 & 3 \end{vmatrix}$$

$$s \times a = (-4 - 6)i - (1 - 4)j + (-3 - 8)k$$

$$s \times a = -10i + 3j - 11k$$

Example 1d

If $\boldsymbol{a} = 2i + 3j - k$, $\boldsymbol{s} = -i + 4j + 2k$, and $\boldsymbol{u} = 2i + j$, find:

d. $\boldsymbol{s} \times \boldsymbol{s}$

$$s \times s = \begin{vmatrix} i & j & k \\ -1 & 4 & 2 \\ -1 & 4 & 2 \end{vmatrix}$$

$$s \times s = i\begin{vmatrix} 4 & 2 \\ 4 & 2 \end{vmatrix} - j\begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix} + k\begin{vmatrix} -1 & 4 \\ -1 & 4 \end{vmatrix}$$

$$s \times s = (8 - 8)i - (-2 - -2)j + (-4 - -4)k$$

$$s \times s = 0i - 0j + 0k$$

$$s \times s = 0$$

Quick Check 1

Given **u** = **i** + 2**j** + **k** and **v** = 3**i** + **j** + 2**k**, find the cross product of **u** × **v**

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{k}$$

$$= (4 - 1)\mathbf{i} - (2 - 3)\mathbf{j} + (1 - 6)\mathbf{k}$$

$$= 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$$

The Cross Product

These properties, and several others, are summarized in the following list.

Algebraic Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space and let c be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Geometric Properties of the Cross Product

The following list gives some other *geometric* properties of the cross product of two vectors.

Geometric Properties of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$

3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.

4. $\|\mathbf{u} \times \mathbf{v}\| =$ area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

Example 2

Find the unit vector that is orthogonal to both,

$\boldsymbol{a} = 2i + 4j - 3k$ and $\boldsymbol{s} = -2i - 3j + 2k$

$$a \times s = \begin{vmatrix} i & j & k \\ 2 & 4 & -3 \\ -2 & -3 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & -3 \\ -3 & 2 \end{vmatrix}(i) - \begin{vmatrix} 2 & -3 \\ -2 & 2 \end{vmatrix}(j) + \begin{vmatrix} 2 & 4 \\ -2 & -3 \end{vmatrix}(k)$$

$$a \times s = (8 - 9)(i) - (4 - 6)(j) + (-6 - -8)(k)$$

$$a \times s = -1i - (-2)(j) + 2k$$

$$a \times s = -1i + 2j + 2k$$ Vector is Orthogonal to a and s

$$\|a \times s\| = \sqrt{(-1)^2 + (2)^2 + (2)^2}$$

$$\|a \times s\| = \sqrt{1 + 4 + 4}$$

$$\|a \times s\| = 3$$

$$\frac{a \times s}{\|a \times s\|} = \frac{-1}{3}i + \frac{2}{3}j + \frac{2}{3}k$$

Quick Check 2

Find a unit vector that is orthogonal to both

u = 3**i** – 4**j** + **k** and **v** = –3**i** + 6**j**.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 1 \\ -3 & 6 & 0 \end{vmatrix} = -6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$$

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-6)^2 + (-3)^2 + 6^2}$$

$$= \sqrt{81}$$

$$= 9$$

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$