9.1 SEQUENCES AND SERIES

- *Use sequence notation to write the terms of a sequence*
- *Use a recursive rule to find terms of a sequence*
- *Use factorial notation*
- *Use sigma notation to represent a sequence*
- *Find the sum of a series*
- *Use sequence and series notation to model and solve real-world problems*

Sequence – (a pattern of numbers); a function whose domain is the set of positive integers. The function values or terms of a sequence are written as $a_1, a_2, a_3, a_4, \ldots$ Rather than using function notation, sequences are usually written with subscripts

A **finite sequence** is a pattern that has a specific number of terms

An **infinite sequence** is a pattern that goes on forever; there is no "last" term

The **general term** (**nth term**) is a "formula" that describes how to find all the specific terms of the pattern.

EXAMPLE 1 – WRITING THE TERMS OF A SEQUENCE

a. Find the first 4 terms of $a_n = 2n + 8$

$Let\ n = 1$	$a_1 = 2(1) + 8$	$a_1 = 10$	
$Let\ n = 2$	$a_2 = 2(2) + 8$	$a_2 = 12$	$10, 12, 14, 16$
$Let\ n = 3$	$a_3 = 2(3) + 8$	$a_3 = 14$	
$Let\ n = 4$	$a_4 = 2(4) + 8$	$a_4 = 16$	

b. Find the first 4 terms of $a_n = \frac{(-1)^n}{n^2}$

$Let\ n = 1$	$a_1 = \frac{(-1)^1}{1^2}$	$a_1 = -1$	$Let\ n = 3$	$a_3 = \frac{(-1)^3}{3^2}$	$a_3 = -\frac{1}{9}$
$Let\ n = 2$	$a_2 = \frac{(-1)^2}{2^2}$	$a_2 = \frac{1}{4}$	$Let\ n = 4$	$a_4 = \frac{(-1)^4}{4^2}$	$a_4 = \frac{1}{16}$

$$-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}$$

EXAMPLE 2 – FINDING THE GENERAL (NTH) TERM

a. Find the general term $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

$$a_1 = \frac{1}{2} \qquad a_2 = \frac{1}{4} \qquad a_3 = \frac{1}{8} \qquad a_4 = \frac{1}{16}$$

$$2^1 = 2 \qquad 2^2 = 4 \qquad 2^3 = 8 \qquad 2^4 = 16$$

$$a_n = \frac{1}{2^n} \qquad a_n = 2^{-n}$$

b. Find the general term $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{6}, \frac{1}{8}, -\frac{1}{10} \ldots$

$$a_1 = -\frac{1}{2} \qquad a_2 = \frac{1}{4} \qquad a_3 = -\frac{1}{6} \qquad a_4 = \frac{1}{8} \qquad a_5 = -\frac{1}{10}$$

$$-\frac{1}{2} = \frac{(-1)^1}{2 \cdot 1} \quad \frac{1}{4} = \frac{(-1)^2}{2 \cdot 2} \quad -\frac{1}{6} = \frac{(-1)^3}{2 \cdot 3}$$

$$a_n = \frac{(-1)^n}{2n}$$

Note that in Ex.1b and Ex. 2b, the signs of the terms alternate between positive and negative.

This is called an alternating sequence and we must include $(-1)^n$ as part of the general term.

FACTORIAL NOTATION:

$$n! = n(n-1)(n-2)(n-3)\cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

$$0! = 1$$

$$1! = 1$$

$$2! = 2 \cdot 1 = 2$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

EXAMPLE 3 – EVALUATING FACTORIALS

a. $\dfrac{5!}{3!2!}$

$$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}$$

$$\frac{5 \cdot 2}{1}$$

$$10$$

b. $\dfrac{3!4!}{2!6!}$

$$\frac{3 \cdot 2! \cdot 4!}{2! \cdot 6 \cdot 5 \cdot 4!}$$

$$\frac{1}{2 \cdot 5}$$

$$\frac{1}{10}$$

Be aware: $2!\,3! \neq 6!$ This is a common mistake!

EXAMPLE 3 – EVALUATING FACTORIALS

c. Find the first 4 terms $\frac{x^n}{(n+1)!}$

$Let\ n = 1 \quad \frac{x^1}{(1+1)!} = \frac{x}{2!} = \frac{x}{2}$

$Let\ n = 2 \quad \frac{x^2}{(2+1)!} = \frac{x^2}{3!} = \frac{x^2}{6}$

$Let\ n = 3 \quad \frac{x^3}{(3+1)!} = \frac{x^3}{4!} = \frac{x^3}{24}$

$Let\ n = 4 \quad \frac{x^4}{(4+1)!} = \frac{x^4}{5!} = \frac{x^4}{120}$

d. $\frac{(n+2)!}{n!}$

$Let\ n = 1 \quad \frac{(1+2)!}{1!} = \frac{3!}{1} = 6$

$6, 12, 20, 30$

RECURSION FORMULAS – USING A RECURSIVE RULE IS ANOTHER WAY THAT WE CAN DEFINE/DESCRIBE A SEQUENCE

- In a recursion formula, the first few terms are listed and then there is a rule for how to find the rest of the terms in the pattern. The subsequent terms are found only by knowing the value of the preceding terms(s).
- This is not good a good method for finding a specific term in the sequence, but rather the recursion rule just tells us how to extend the pattern.

EXAMPLE 4 – RECURSIVE SEQUENCE

Find the first five terms of the sequence defined recursively as

$$a_1 = 3, \quad a_k = 2a_{k-1} + 1, \quad where\ \ k \geq 2$$

3, 7, 15, 31, 63

$a_1 = 3$

$a_2 = 2a_1 + 1$ $\quad a_2 = 2(3) + 1$ $\quad a_2 = 7$

$a_3 = 2a_2 + 1$ $\quad a_3 = 2(7) + 1$ $\quad a_3 = 15$

$a_4 = 2a_3 + 1$ $\quad a_4 = 2(15) + 1$ $\quad a_4 = 31$

$a_5 = 2a_4 + 1$ $\quad a_5 = 2(31) + 1$ $\quad a_5 = 63$

SERIES -
THE SUM OF THE TERMS OF A SEQUENCE

The sum of a finite sequence is called a **finite series** or a **partial sum**

The sum of an infinite sequence is called an **infinite series**
(not all infinite sequences will have a sum)

The capital Greek letter (sigma) Σ is a shorthand ay to represent a series.

Σ means "the sum of…" We call this **sigma notation** or **summation notation.**

For example, $\sum_{n=1}^{5} n^2$ is read as "the sum, as *n* goes from 1 to 5, of *n* squared"

$\sum_{n=1}^{\infty} n^2$ is read as " the sum, , as *n* goes from 1 to infinity, of *n* squared"

The letter *n* is called the **index of summation** (and other letters may be used instead of *n*)

EXAMPLE 5 – SIGMA NOTATION

Write the series using sigma notation.

a. $4 + 8 + 16 + 32 + 64$

$\sum_{n=1}^{5} ??$

$a_1 = 4 \qquad a_1 = 2^{1+1}$

$a_2 = 8 \qquad a_2 = 2^{2+1}$

$a_3 = 16 \qquad a_3 = 2^{3+1}$

$a_4 = 32 \qquad a_4 = 2^{4+1}$

$$\sum_{n=1}^{5} 2^{n+1}$$

b. $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \cdots$

$\sum_{k=1}^{\infty} (-1)^{k+1}$

$a_1 = 1 \qquad a_1 = 1$

$a_2 = \frac{1}{2} \qquad a_2 = \frac{1}{2 \cdot 1}$

$a_3 = \frac{1}{6} \qquad a_3 = \frac{1}{3 \cdot 2 \cdot 1}$

$a_4 = \frac{1}{24} \qquad a_4 = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1}$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{k!}\right)$$

EXAMPLE 6 – EVALUATE THE FINITE SERIES

Write the terms, then simplify.

$$\sum_{j=3}^{7}(4j-1)$$

$a_3 = 4(3) - 1$ $\qquad a_3 = 11$

$a_4 = 4(4) - 1$ $\qquad a_4 = 15$

$a_5 = 4(5) - 1$ $\qquad a_5 = 19$

$a_6 = 4(6) - 1$ $\qquad a_6 = 23$

$a_7 = 4(7) - 1$ $\qquad a_7 = 27$

$$\sum_{j=3}^{7}(4j-1) = 11 + 15 + 19 + 23 + 27 \quad = 95$$

EXAMPLE 7 – PARTIAL SUM

Find the third partial sum, then find the sum of $\sum_{n=1}^{\infty} \frac{4}{10^n}$

Find the third partial sum means to find:

$$\sum_{n=1}^{3} \frac{4}{10^n} = \frac{4}{10^1} + \frac{4}{10^2} + \frac{4}{10^3}$$

$$= \frac{4}{10} + \frac{4}{100} + \frac{4}{1000}$$

$$= 0.4 + 0.04 + 0.004$$

$$= 0.444$$

So to find: $\sum_{n=1}^{\infty} \frac{4}{10^n}$ Let $x = 0.444\ldots$

$$10x = 4.444\ldots$$

$$-x = -0.444\ldots$$

$$9x = 4$$

$$x = \frac{4}{9}$$

$$\sum_{n=1}^{\infty} \frac{4}{10^n} = \frac{4}{9}$$

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#9, 11, 15, 19, 23, 25, 27, 31, 37, 39, 45,

49-55 (odd), 59-71 (odd), 79, 83, 85, 89-97 (odd)