9.5 The Binomial Theorem

Objectives

- Use the Binomial Theorem to calculate binomial coefficients.
- Use Pascal's Triangle to calculate binomial coefficients.
- Use binomial coefficients to write binomial expansions.

Binomial Coefficients

Recall that a *binomial* is a polynomial that has two terms.

In this section, you will study a formula that provides a quick method of raising a binomial to a power. To begin, look at the expansion of $(x + y)^n$ for several values of n.

$(x + y)^0 = 1$

$(x + y)^1 = x + y$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

Binomial Coefficients

There are several observations you can make about these expansions.

1. In each expansion, there are $n + 1$ terms.

2. In each expansion, x and y have symmetrical roles. The powers of x decrease by 1 in successive terms, whereas the powers of y increase by 1.

3. The sum of the powers of each term is n. For instance, in the expansion of $(x + y)^5$, the sum of the powers of each term is 5.

$$(x + y)^5 = x^5 + 5\underbrace{x^4y^1}_{4+1=5} + 10\underbrace{x^3y^2}_{3+2=5} + 10x^2y^3 + 5x^1y^4 + y^5$$

Binomial Coefficients

4. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients.** To find them, you can use the **Binomial Theorem.**

The Binomial Theorem

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r\, x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

The symbol $\binom{n}{r}$ is often used in place of ${}_nC_r$ to denote binomial coefficients.

Example 1 – *Finding Binomial Coefficients*

Find each binomial coefficient.

a. ${}_8C_2$ **b.** $\binom{10}{3}$ **c.** ${}_7C_0$ **d.** $\binom{8}{8}$

Solution:

a.

$$
\begin{aligned}
{}_8C_2 &= \frac{8!}{6! \cdot 2!} \\
&= \frac{(8 \cdot 7) \cdot \cancel{6!}}{\cancel{6!} \cdot 2!} \\
&= \frac{8 \cdot 7}{2 \cdot 1} \\
&= 28
\end{aligned}
$$

b.

$$
\begin{aligned}
\binom{10}{3} &= \frac{10!}{7! \cdot 3!} \\
&= \frac{(10 \cdot 9 \cdot 8) \cdot \cancel{7!}}{\cancel{7!} \cdot 3!} \\
&= \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} \\
&= 120
\end{aligned}
$$

Example 1 – *Solution*

cont'd

c. $${}_7C_0 = \frac{\cancel{7!}}{\cancel{7!} \cdot 0!}$$

$$= 1$$

d. $$\binom{8}{8} = \frac{\cancel{8!}}{0! \cdot \cancel{8!}}$$

$$= 1$$

Binomial Coefficients

When $r \neq 0$ and $r \neq n$, as in parts (a) and (b) above, there is a pattern for evaluating binomial coefficients that works because there will always be factorial terms that divide out from the expression.

$$ {}_8C_2 = \frac{\overbrace{8 \cdot 7}^{\text{2 factors}}}{\underbrace{2 \cdot 1}_{\text{2 factors}}} \quad \text{and} \quad \binom{10}{3} = \frac{\overbrace{10 \cdot 9 \cdot 8}^{\text{3 factors}}}{\underbrace{3 \cdot 2 \cdot 1}_{\text{3 factors}}} $$

In general, it is true that

$$ {}_nC_r = {}_nC_{n-r}. $$

This shows the symmetric property of binomial coefficients.

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1 $4 + 6 = 10$

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1 $15 + 6 = 21$

The first and last numbers in each row of Pascal's Triangle are 1. Every other number in each row is formed by adding the two numbers immediately above the number.

Pascal's Triangle

Pascal noticed that numbers in this triangle are precisely the same numbers that are the coefficients of binomial expansions, as follows.

$$(x + y)^0 = 1 \qquad \text{0th row}$$

$$(x + y)^1 = 1x + 1y \qquad \text{1st row}$$

$$(x + y)^2 = 1x^2 + 2xy + 1y^2 \qquad \text{2nd row}$$

$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 \qquad \text{3rd row}$$

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \qquad \vdots$$

$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

$$(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$

$$(x + y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7$$

Example 2a – *Use Pascal's Triangle*

a. $(x+2)^3$ 1 3 3 1

x^3 x^2 x^1 x^0

2^0 2^1 2^2 2^3

$$1 \cdot 1 \cdot x^3 + 3 \cdot 2 \cdot x^2 + 3 \cdot 4 \cdot x^1 + 1 \cdot 1 \cdot 8$$

$$x^3 + 6x^2 + 12x + 8$$

Example 2b – *Use Pascal's Triangle*

b. $(x-3)^4$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$x^4 \quad x^3 \quad x^2 \quad x^1 \quad x^0$$

$$(-3)^0 \quad (-3)^1 \quad (-3)^2 \quad (-3)^3 \quad (-3)^4$$

$$1 \cdot 1 \cdot x^4 + 4 \cdot -3 \cdot x^3 + 6 \cdot 9 \cdot x^2 + 4 \cdot -27 \cdot x^1 + 1 \cdot 1 \cdot 81$$

$$x^4 - 12x^3 + 54x^2 - 108x + 81$$

Example 2c – *Use Pascal's Triangle*

c. $(a-2b)^5$

1	5	10	10	5	1
a^5	a^4	a^3	a^2	a^1	a^0
$(-2b)^0$	$(-2b)^1$	$(-2b)^2$	$(-2b)^3$	$(-2b)^4$	$(-2b)^5$

$$a^5 - 10a^4b + 40a^3b^2 - 80a^2b^3 + 80ab^4 - 32b^5$$

Binomial Expansions

Sometimes you will need to find a specific term in a binomial expansion.

The coefficient of the term $x^{n-k}y^{k}$ in the expansion of $(x+y)^{n}$ is

$$\binom{n}{k} = {}_{n}C_{k} = C(n,k) = \frac{n!}{k!(n-k)!}$$

Example 3a - Find each specified term

a. the 3rd term of $(x-4)^6$

$$(x+y)^n = \binom{n}{k} \cdot x^{n-k} \cdot y^k$$

$x = x$

$y = -4$

$n = 6$

$k = 3 - 1 = 2$

$$(x-4)^6 = \binom{6}{2} \cdot x^{6-2} \cdot (-4)^2$$

$$(x-4)^6 = 15 \cdot x^4 \cdot 16$$

$$(x-4)^6 = 240x^4$$

Example 3b - Find each specified term

b. the seventh term of $(2x+3y)^{13}$

$$(x+y)^n = \binom{n}{k} \cdot x^{n-k} \cdot y^k$$

$x = 2x$

$y = 3y$

$n = 13$

$k = 7 - 1 = 6$

$$(2x+3y)^{13} = \binom{13}{6} \cdot (2x)^{13-6} \cdot (3y)^6$$

$$(2x+3y)^{13} = 1716 \cdot 128x^7 \cdot 729y^6$$

$$(2x+3y)^{13} = 160{,}123{,}392x^7y^6$$

Example 3c - Find each specified term

b. the fifth term of $(3x-y)^{15}$

$$(x+y)^n = \binom{n}{k} \cdot x^{n-k} \cdot y^k$$

$x = 3x$

$y = -y$

$n = 15$

$k = 5 - 1 = 4$

$$(3x-y)^{15} = \binom{15}{4} \cdot (3x)^{15-4} \cdot (-y)^4$$

$$(3x-y)^{15} = 1365 \cdot 177{,}147x^{11} \cdot y^4$$

$$(3x-y)^{15} = 241{,}805{,}655x^{11}y^4$$