9.6 Counting Principles

Objectives

- Solve simple counting problems.
- Use the Fundamental Counting Principle to solve counting problems.
- Use permutations to solve counting problems.
- Use combinations to solve counting problems.

Example 1 – *Selecting Pairs of Numbers at Random*

You place eight pieces of paper, numbered from 1 to 8, in a box. You draw one piece of paper at random from the box, record its number, and *replace* the paper in the box.

Then, you draw a second piece of paper at random from the box and record its number.

Finally, you add the two numbers. How many different ways can you obtain a sum of 12?

Example 1 – *Solution*

To solve this problem, count the different ways to obtain sum of 12 using two numbers from 1 to 8.

First number	4	5	6	7	8
Second number	8	7	6	5	4

So, a sum of 12 can occur in five different ways.

Example 1 b

You repeat the experiment again. However, this time you *do not* replace the first paper before drawing the second paper.

Now how many different ways can you obtain a sum of 12?

To solve this problem, count the different ways to obtain sum of 12 using two numbers from 1 to 8.

First number	4	5	7	8
Second number	8	7	5	4

So, a sum of 12 can occur in four different ways.

The Fundamental Counting Principle

The most important of these is the **Fundamental Counting Principle.**

Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.

The Fundamental Counting Principle can be extended to three or more events. For instance, the number of ways that three events E_1, E_2, and E_3 can occur is $m_1 \cdot m_2 \cdot m_3$.

Example 2 – *Using the Fundamental Counting Principle*

How many different pairs of letters from the English alphabet are possible?

Solution:

There are two events in this situation.

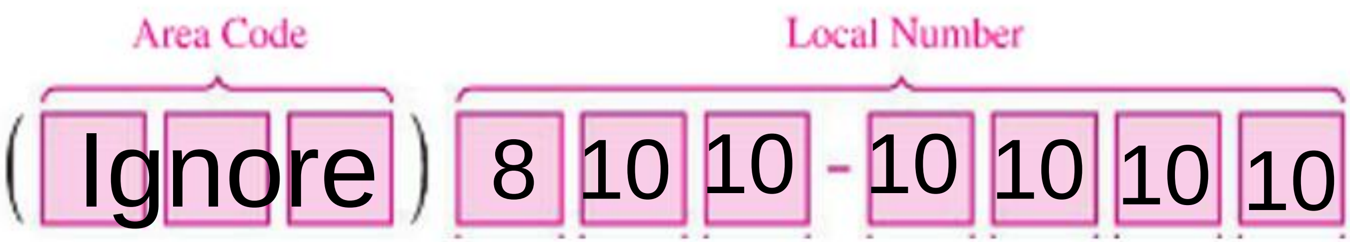
The first event is the choice of the first letter, and the second event is the choice of the second letter.

Because the English alphabet contains 26 letters, it follows that the number of two-letter pairs is

$$26 \cdot 26 = 676.$$

Example 3a

a. Telephone numbers in the U.S. have 10 digits. The first three digits are the area code and the next seven digits are the local telephone number. How many different local telephone numbers are possible within each area code? (A local number cannot begin with a 0 or 1.)



8,000,000

Example 3b

b. A product's catalog number is made up of two letters followed by a four-digit number. How many different catalog numbers are possible?

26	26	10	10	10	10

6,760,000

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that *n* elements can be arranged (in order).

An ordering of *n* elements is called a **permutation** of the elements.

Definition of Permutation

A **permutation** of *n* different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

Example 4 – *Finding the Number of Permutations*

How many permutations of the letters A, B, C, D, E, and F are possible?

Solution:

Consider the following reasoning.

First position: Any of the *six* letters

Second position: Any of the remaining *five* letters

Third position: Any of the remaining *four* letters

Fourth position: Any of the remaining *three* letters

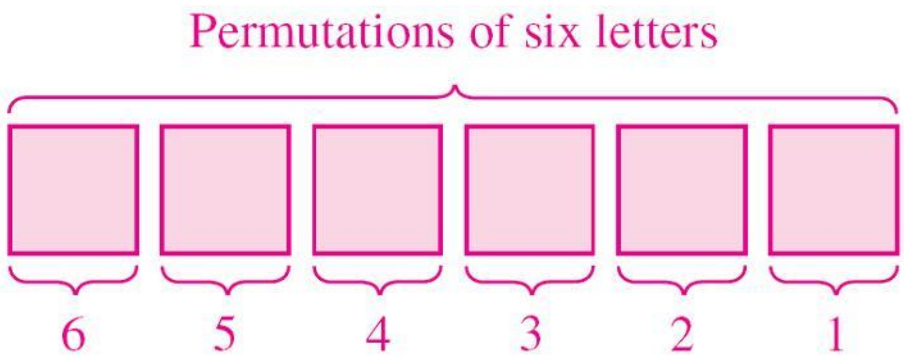
Fifth position: Either of the remaining *two* letters

Sixth position: The *one* remaining letter

Example 4 – *Solution*

cont'd

So, the numbers of choices for the six positions are as follows.



The total number of permutations of the six letters is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 720.$$

Permutations

Number of Permutations of *n* Elements

The number of permutations of n elements is

$$n \cdot (n-1) \cdot \cdot \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways that n elements can be ordered.

It is useful, on occasion, to order a *subset* of a collection of elements rather than the entire collection.

For example, you might want to order r elements out of a collection of n elements.

Example 5

a. Seven horses are running in a race. In how many different ways can these horses come in first, second, and third?

7	6	5

210

b. A club has six members. In how many ways can there be a president and a vice-president?

6	5

30

Permutations

Such an ordering is called a **permutation of *n* elements taken *r* at a time.**

Permutations of *n* Elements Taken *r* at a Time

The number of permutations of n elements taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdot\cdot\cdot(n-r+1).$$

Remember that for permutations, order is important. So, to find the possible permutations of the letters A, B, C, and D taken three at a time, you count the permutations (A, B, D) and (B, A, D) as different because the *order* of the elements is different.

Example 6

Rework Example 5 using the permutation formula.

a. Seven horses are running in a race. In how many different ways can these horses come in first, second, and third?

$$_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 210$$

b. A club has six members. In how many ways can there be a president and a vice-president?

$$_6P_2 = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6 \cdot 5 \cdot 4!}{4!} = 30$$

Permutations

However, not all of these arrangements would be *distinguishable* because there are two A's in the list.

To find the number of distinguishable permutations, you can use the following formula.

Distinguishable Permutations

Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \cdot \cdot \cdot + n_k.$$

Then the number of **distinguishable permutations** of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdot \cdot \cdot \cdot n_k!}.$$

Example 7

How many distinguishable permutations are there for the letters in the following words?

a. AGREEABLE

$n = 9$

A $= 2$

E $= 3$

$$\frac{9!}{2!\,3!}$$

$$= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 2 \cdot 1}$$

$$= 30{,}240$$

b. BOOKKEEPER

$n = 10$

O $= 2$

K $= 2$

E $= 3$

$$\frac{10!}{2!\,2!\,3!}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 2 \cdot 1 \cdot 2 \cdot 1}$$

$$= 151{,}200$$

Combinations

Combinations of *n* Elements Taken *r* at a Time

The number of combinations of n elements taken r at a time is

$$_nC_r = \frac{n!}{(n - r)!r!}$$

which is equivalent to $_nC_r = \frac{_nP_r}{r!}$.

Example 8

a. In three-card poker, each player is dealt three cards from a standard deck of 52 cards.

How many three-card poker hands are possible? $_{52}C_3 = 22{,}100$

How many 5-card poker hands are possible? $_{52}C_5 = 2{,}598{,}960$

b. You are forming a 12-member swim team from 10 girls and 15 boys. The team must consist of 6 boys and six girls. How many different 12-member teams are possible?

$$(girls) \cdot (boys) = {}_{10}C_6 \cdot {}_{15}C_6 = (210) \cdot (5005) = 1{,}051{,}050$$

c. Using the information from Example 7b, how many 12-member teams could be formed if the team must have 8 girls and four boys?

$$(girls) \cdot (boys) = {}_{10}C_8 \cdot {}_{15}C_4 = (45) \cdot (1365) = 61{,}425$$

Combinations

When solving problems involving counting principles, you need to be able to distinguish among the various counting principles in order to determine which is necessary to solve the problem correctly.

To do this, ask yourself the following questions.

1. Is the order of the elements important? *Permutation*

2. Are the chosen elements a subset of a larger set in which order is not important? *Combination*

3. Does the problem involve two or more separate events? *Fundamental Counting Principle*